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Uniform formulas for infinite antenna current

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A uniform asymptotic expansion of the time-harmonic antenna current with large parameter $\ln [2kz/(ka)^2]$ is given. As usual, the expansion is based on Hallén's filament integral and is restricted by the condition $z \gg a$ but is otherwise uniform in kz. The use of averaging leads to the vanishing of the second term of the asymptotic series. Two different averaging procedures are used to derive two accurate formulas: The first one, which is very simple, is useful for practical applications. The second formula is of theoretical interest. An improved asymptotic series for large kz is derived to relate the averaging results to existing asymptotic results. Comparison of these two formulas and the exact antenna integral demonstrates that these formulas are more accurate than any other existing formulas for infinite cylindrical antenna current. The Hallén filament integral is found to approximate closely the exact integral even when the condition $z \gg a$ is violated. Discrepancies in the literature are discussed.

1. INTRODUCTION

It is interesting to note that early authors hesitated to consider an infinite antenna. *Hallén* [1948] had to justify its consideration by saying, "An infinite antenna does not exist, but the behavior of the outgoing traveling current waves on an antenna of finite length must be the same as if it were infinite. . . . " The need for good infinite antenna solutions is universal: *Hallén* [1962] and *King* [1956] used an infinite antenna solution to illustrate the difference between the antenna and the transmission line solutions. In antenna handbooks [*Rispin and Chang*, 1988; *King and Smith*, 1981; *Vainshtein*, 1969] the finite antenna solution is constructed by means of an infinite antenna solution.

The earliest treatment of an infinite cylindrical antenna was given by *Stratton and Chu* [1941]. Other important works on antenna admittance and feedgap problems were given by *Brillouin* [1943], *Infeld* [1947], *Albert and Synge* [1948], *Synge* [1948], *Wu and King* [1959], *Duncan* [1962], *Chen and Keller* [1962], *Fante* [1966], and *Latham and Lee* [1969]. Two Russian articles treated an infinite length antenna [*Vainshtein*, 1961*b*] and a finite length antenna [*Vainshtein and Fock*, 1968]. The first correct asymptotic solution valid to the leading (zero) order was obtained by *Vladimirskii* [1944], and to the first order a little later by *Hallén* [1948]. Numerous other authors [*Vainshtein*, 1959, 1961; *Kuehl*, 1960; *Wu*, 1961; *Kunz*, 1963] have discussed asymptotic expansions of antenna current.

Papas [1948] obtained an asymptotic expansion of the magnetic field away from the antenna for an infinite cylindrical antenna and correctly conjectured that the form of the solution for the antenna current would be similar. *Northover* [1958] discussed asymptotic solutions for the magnetic field and the current of an infinite antenna, and pointed out a difference of a factor of 2 when comparing the magnetic field away from the antenna and the antenna current. *Latham and Lee* [1970] obtained the time domain magnetic field and showed graphically a factor of 2 transition from the antenna current to the magnetic field away from the antenna.

The first uniform formula for infinite antenna current was obtained in the work by *Vainshtein* [1961] in the Russian literature. *Shen et al.* [1968] also gave an interpolation formula, which will be referred to as the SWK interpolating formula. Weinstein's formula has been used in the antenna literature for constructing the scattering antenna current excited by a grazing incident plane wave [*Chang et al.*, 1978].

Averaging has been shown to be a powerful way to construct uniform formulas [*Chen*, 1983, 1987; *Chen and Warne*, 1991]. The gist of averaging has been discussed in the treatment of time domain antenna current [*Chen and Warne*, 1991]. Averag-

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ing is used here to derive two formulas for time-harmonic current: One formula is very simple; the other shows the connection to Weinstein's formula and the more general solution for loaded antennas [*Chen and Warne*, 1992]. These uniform formulas are not only the most accurate formulas, but they also can be used to understand the small differences between the antenna solution (in the limit of vanishing ka) for moderate length and transmission line solutions. *Pocklington*'s [1897] sinusoidal distribution is, of course, basically a transmission line solution. The great difference between the antenna and transmission line solutions only applies to long antennas. The time-harmonic and time domain behavior of a long antenna in a dissipative medium, however, also resembles a transmission line solution [*Chen*, 1987].

Finally, some explanations on the asymptotic solution of infinite antenna current are helpful in understanding the literature, since the infinite antenna is such a basic problem in electromagnetic theory. The main objective, however, is to derive accurate formulas for the time-harmonic antenna current and show the accuracy of these new formulas.

2. DERIVATION OF A SIMPLE FORMULA FOR THE TIME-HARMONIC ANTENNA CURRENT

The time-harmonic current on an infinite tubular antenna (with radius a) driven at a delta gap by a voltage $-V$ is (with $e^{-i\omega t}$ time dependence) [*Hallén*, 1948, 1962; *Wu*, 1969]

$$I(z) = \frac{2ikV}{\eta_0} \cdot \int_C \frac{e^{i\xi z}\, d\xi}{(k^2 - \xi^2)\pi i J_0[a(k^2 - \xi^2)^{1/2}] H_0^{(1)}[a(k^2 - \xi^2)^{1/2}]}, \tag{1}$$

where $\eta_0 = (\mu_0/\varepsilon_0)^{1/2}$ is the intrinsic impedance of the medium and $k = \omega/c$. The usual contour of integration C is indented below the branch point at k but indented above the branch point at $-k$ (Figure 1). When the contour is so chosen $I(z)$ in (1) gives the sum of the exterior and interior currents. Because the current (1) is even with respect to z, we can confine our attention to $z > 0$. The interior current arises from the residue contribution of the poles (zeros of J_0) lying on the positive imaginary axis. When the exterior antenna current is of primary concern as in this paper, we do not include the residue contributions of all poles on the imaginary axis [*Hallén*, 1962].

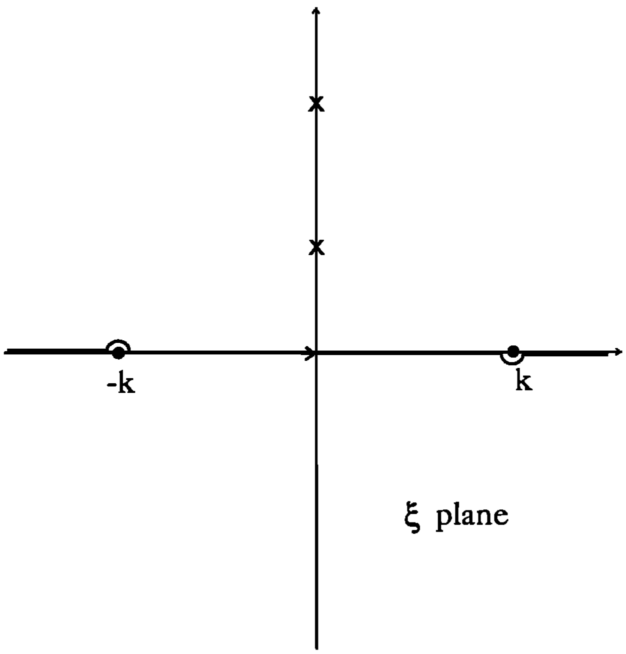

Fig. 1. The contour of integration is indented below the branch point at k but indented above the branch point at $-k$. When the contour is deformed to the upper half plane, the interior current arises from the residue contribution of the zeros of $J_0[a(k^2 - \xi^2)^{1/2}]$ lying on the positive imaginary axis (shown as "x"). When the exterior antenna current is the subject of study, we do not include these residue contributions.

To obtain an asymptotic solution for large z, it is permissible to approximate the Bessel functions by the leading terms of their small-argument expansions [*Vladimirskii*, 1944]. However, it should be emphasized that Hallén gave great significance to such an expanded form by interpretating it as a simple physical space kernel. The solution obtained can be used for wires of general cross section and is also free from the logarithmic singularity associated with the gap capacitance. Any realistic feedgap can be defined approximately as the difference between this idealized configuration and the actual configuration, at least for thin antennas. An example of such a treatment can be found in the discussion of effective impedance of a bolt load on a narrow aperture [*Warne and Chen*, 1992]. Another idealized configuration was treated by *Latham and Lee* [1969].

After the incorporation of the small-argument expansions, (1) is reduced to [*Hallén*, 1948; *Vladimirskii*, 1944]

$$I(z) \sim \frac{2ikV}{\eta_0} \cdot \int_C \frac{e^{i\xi z}\, d\xi}{(k^2-\xi^2)\left[-2\ln\left(\frac{a}{2}\right) - \ln(k^2-\xi^2) - 2\gamma + i\pi\right]} \tag{2}$$

Equation (2) can be written as

$$I(z) \sim \frac{V}{\eta_0}\, e^{ikz} G(\beta_p, kz), \tag{3}$$

where

$$G(\beta_p, kz) = -i \int_C \frac{e^{i\chi kz}\, d\chi}{\chi\left(1+\frac{\chi}{2}\right)\left(\beta_p - \ln\left[\chi\left(1+\frac{\chi}{2}\right)\right]\right)}, \tag{4}$$

with

$$\xi = k(1+\chi),\ \beta_p = -2\ln(ka) - 2\gamma + \ln 2. \tag{5}$$

Because of the small-argument expansion of the Bessel functions, extraneous poles located at $\chi = -1 \pm [2e^{\beta_p} + 1]^{1/2}$ have been introduced [*Hallén*, 1962; *Wu*, 1969]. Here we do not go into the discussion of this matter but simply note that the contour C is to be above these poles for $z > 0$. Thus

$$G(\beta_p, kz) = i\int_0^\infty \frac{d\eta e^{-\eta kz}}{\eta\left(1+i\frac{\eta}{2}\right)}\left[\frac{1}{\beta_p - \ln\left[\eta\left(1+i\frac{\eta}{2}\right)\right] + i\frac{3\pi}{2}} - \frac{1}{\beta_p - \ln\left[\eta\left(1+i\frac{\eta}{2}\right)\right] - i\frac{\pi}{2}}\right]. \tag{6}$$

Equation (6) can be simplified if we change the signs of the denominator terms $\ln(1 + i\eta/2)$. The approximation in general requires $|\beta_p| \gg 1$ (Note furthermore when kz is large $\ln(1 + i\eta/2) \to 0$.) so that these terms in the denominator of (6) are small anyway. Thus (6) becomes

$$G(\beta_p, kz) = i\int_0^\infty d\eta e^{-\eta kz}\left(\frac{1}{\eta} - \frac{i/2}{1+i\frac{\eta}{2}}\right) \cdot \left[\frac{1}{\beta_p - \ln\eta + \ln\left(1+i\frac{\eta}{2}\right) + i\frac{3\pi}{2}} - \frac{1}{\beta_p - \ln\eta + \ln\left(1+i\frac{\eta}{2}\right) - i\frac{\pi}{2}}\right] \tag{7}$$

A subsequent integration by parts leads to

$$G(\beta_p, kz) = ikz\int_0^\infty d\eta e^{-\eta kz} \cdot \ln\left[\frac{\beta_p - \ln(\eta) + \ln\left(1+i\frac{\eta}{2}\right) - i\frac{\pi}{2}}{\beta_p - \ln(\eta) + \ln\left(1+i\frac{\eta}{2}\right) + i\frac{3\pi}{2}}\right]. \tag{8}$$

Two distinct averaging procedures will be used below. First, a simple averaging of $\ln(\eta)$ over the weight $e^{-\eta kz}$ is defined as

$$\frac{\int_0^\infty d\eta \ln(\eta) e^{-\eta kz}}{\int_0^\infty d\eta\, e^{-\eta kz}} = -\ln(kz) - \gamma. \tag{9}$$

Using (9) in (8) and the exponential of (9) for η in $\ln(1 + i\eta/2)$, we obtain

$$G(\beta_p, kz) \sim i\ln\left[\frac{\beta_p + \gamma - \ln 2 + \ln(2kz + ie^{-\gamma}) - i\frac{\pi}{2}}{\beta_p + \gamma - \ln 2 + \ln(2kz + ie^{-\gamma}) + i\frac{3\pi}{2}}\right]$$

$$= i\ln\left[\frac{-2\ln(ka) - \gamma + \ln(2kz + ie^{-\gamma}) - i\frac{\pi}{2}}{-2\ln(ka) - \gamma + \ln(2kz + ie^{-\gamma}) + i\frac{3\pi}{2}}\right]. \tag{10}$$

3. DERIVATION OF AN ALTERNATIVE FORMULA

An alternative averaging procedure can be defined as

$$\frac{\displaystyle\int_0^\infty d\eta e^{-\eta kz}\left[\ln(\eta) - \ln\left(1 + i\frac{\eta}{2}\right)\right]}{\displaystyle\int_0^\infty d\eta e^{-\eta kz}}$$

$$= -\ln(kz) - \gamma - e^{-i2kz}E_1(-i2kz). \qquad (11)$$

Here E_1 is an exponential integral. Use of (11) in (8) gives

$$G(\beta_p, kz) \sim i \ln\left[\frac{\Omega(z) - i\pi}{\Omega(z) + i\pi}\right], \qquad (12)$$

where

$$\Omega(z) = \ln\frac{2kz}{(ka)^2} - \gamma + i\frac{\pi}{2} + e^{-i2kz}E_1(-i2kz). \qquad (13)$$

Note that E_1 can be expressed as sine and cosine integrals:

$$E_1(-i2kz) = -Ci(2kz) - i\,si(2kz), \qquad (14)$$

where $si(2kz) = Si(2kz) - \pi/2$.

4. SUMMARY OF EXISTING FORMULAS FOR ANTENNA CURRENT

Frequently used formulas in the literature are compared to the new formulas (10) and (12). In the next section they are compared to the numerical integration values of the exact solution (1) and the Hallén filament approximation (6).

The Hallén first-order formula (good for two terms) is the earliest first-order asymptotic result for large parameter ln $[2kz/(ka)^2]$. Let Ω be given by (Ω is asymptotically equivalent to $\Omega(z)$ as $kz \to \infty$.)

$$\Omega = \ln\frac{2kz}{(ka)^2} - \gamma + i\frac{\pi}{2}. \qquad (15)$$

Then, Hallén's formula (with $G(\beta_p, kz)$ defined in (3)) is [*Hallén*, 1948]

$$G(\beta_p, kz) \sim \frac{2\pi}{\Omega}. \qquad (16)$$

Equation (16) is also discussed by *Vainshtein* [1959].

Two approximate formulas for infinite cylindrical antennas are described by *Shen et al.* [1968]. A truncation method is used to derive an approximate formula for large kz:

$$G(\beta_p, kz) \sim i \ln\left[\frac{\Omega - i\pi}{\Omega + i\pi}\right]. \qquad (17)$$

This formula is also derived in the limit as $kz \to \infty$ by averaging [*Chen*, 1983] and will be shown later to be a first-order asymptotic solution for a different choice of a large parameter. Equation (17) will be referred to as the SWK (Shen, Wu, and King) formula. The second formula by *Shen et al.* [1968] is obtained by interpolating (17) and the antenna input admittance derived by *Wu* [1961]:

$$G(\beta_p, kz) \sim i \cdot \ln\left[\frac{-2\ln(ka) - \gamma + \ln\{kz + [(kz)^2 + e^{-2\gamma}]^{1/2}\} - i\dfrac{\pi}{2}}{-2\ln(ka) - \gamma + \ln\{kz + [(kz)^2 + e^{-2\gamma}]^{1/2}\} + i\dfrac{3\pi}{2}}\right]. \qquad (18)$$

Equation (18) is the SWK interpolating formula. Note that this formula has been used for constructing finite antenna solutions [*Rispin and Chang*, 1988; *King and Smith*, 1981].

Finally, (12) with $\Omega(z) \gg 1$ gives

$$G(\beta_p, kz) \sim \frac{2\pi}{\Omega(z)}. \qquad (19)$$

Equation (19) was previously derived by *Vainshtein* [1961*a*] using a variational technique and has been used widely for constructing finite antenna solutions [*Rispin and Chang*, 1988]. This result could have been obtained by applying averaging directly on (4). (First, ln $(1 + \chi/2)$ is neglected, and then ln (χ) is replaced by its average defined over the loop integral with $\{e^{i\chi kz}/[\chi(1 + \chi/2)]\}$ as the weight.)

When $kz \to \infty$, (17) can be shown to be

$$G(\beta_p, kz) \sim \frac{2\pi}{\Omega}\left(1 - \frac{\pi^2}{3\Omega^2}\right). \qquad (20)$$

Chen and Wu [1969] observed that the expansion of an identical formula is accurate to the leading order in $1/\Omega$, but actually, the leading term of (20) is good to the first order. Equations (10), (12), (17), and (19) reduce to (16) as $kz \to \infty$.

When $kz \to 0$, (10) and (12) with (3) become

$$\frac{I(0)}{V} \sim \frac{i}{\eta_0} \ln \left[\frac{-\ln (ka) - \gamma}{-\ln (ka) - \gamma + i\pi} \right] \tag{21}$$

and (19) approaches

$$\frac{I(0)}{V} \sim \frac{\pi}{\eta_0} \frac{1}{\left[-\ln (ka) - \gamma + i \dfrac{\pi}{2} \right]} . \tag{22}$$

Note that when ka is very small ($-\ln (ka) \gg 1$), (21) can be expanded to yield (22); when ka is not very small, (21) gives more accurate numerical values. Equation (22) is the antenna input admittance derived by *Tai* [1955], *Hallén* [1962], *Vainshtein* [1959] and *Borgnis and Papas* [1955]. Note that as $ka \to 0$, (18) becomes

$$\frac{I(0)}{V} \sim \frac{i}{\eta_0} \ln \left[\frac{-\ln (ka) - \gamma + i \dfrac{\pi}{4} - i \dfrac{\pi}{2}}{-\ln (ka) - \gamma + i \dfrac{\pi}{4} + i \dfrac{\pi}{2}} \right]$$

$$\sim \frac{\pi}{\eta_0} \frac{1}{\left[-\ln (ka) - \gamma + i \dfrac{\pi}{4} \right]} . \tag{23}$$

It is not surprising that (23) differs from the correct result (22) because only the conductance is used in the interpolation.

5. NUMERICAL RESULTS

Numerical integration of G, defined by (3), for the exact solution (1) and the Hallén filament approximation (6) is facilitated by subtracting the left-hand side of (24) below and then adding its explicit form (the right-hand side of (24)). This facilitating slowly convergent integral is

$$G_0(\beta_p, kz) = i \int_0^{(kz)^{-1}} \frac{d\eta}{\eta \left(1 + i \dfrac{\eta}{2} \right)} \cdot \left[\frac{1}{\beta_p - \ln \eta + \ln \left(1 + i \dfrac{\eta}{2} \right) + i \dfrac{3\pi}{2}} - \frac{1}{\beta_p - \ln \eta + \ln \left(1 + i \dfrac{\eta}{2} \right) - i \dfrac{\pi}{2}} \right] = i \ln \left[\frac{A - \dfrac{3}{2}}{A + \dfrac{1}{2}} \right], \tag{24}$$

where

$$A = \frac{1}{\pi} \arctan \left(\frac{1}{2kz} \right) + 1 - i \frac{2}{\pi} \left[\frac{1}{4} \ln (1 + 4k^2z^2) - \ln (ka) - \gamma \right]. \tag{25}$$

Table 1 shows a comparison of the exact solution (1) and (3), the Hallén filament approximation (6), the simple averaging formula (10), the alternative averaging formula (12), the SWK formula (17), the Hallén first-order formula (16), the Weinstein formula (19), and the SWK interpolating formula (18) for various ka and kz values. Numerical values are determined to greater accuracies than those shown in the table. Of particular interest is that the Hallén filament approximation is in agreement with the exact solution; even when $ka = kz = 0.1$, the real part of G has a 0.3% error and the imaginary part of G has an 8% error. This accuracy is consistent with the time domain case discussed by *Chen et al.* [1992]. The simple averaging, the alternative averaging, and the Weinstein formula are found to be comparable in accuracy. The SWK interpolating formula has somewhat larger errors because, as shown in (23), the imaginary part of the antenna input admittance is not used in the derivation of the formula. SWK and the Hallén first-order formulas are derived for $kz \to \infty$; however, remarkably, when $kz \geq 0.5$, both the real and imaginary parts are quite accurate. As expected, the SWK formula is slightly more accurate than the Hallén first-order formula, especially when ka is not very small.

Figures 2*a* and 2*b* show, for $ka = 0.5$ and $ka/10 \leq kz \leq 1000$, the real and imaginary parts of G, respectively, for all eight entries discussed in Table 1. Note that even at this large ka value (for realistic antennas) the Hallén filament approximation has only 1% error in the real part when $kz = ka/10$; the imaginary part given by the dotted curve, however, has a large (55%) error. A realistic antenna feedgap is probably of the order of ka in width; in such a case the discrepancies of the current obtained from

TABLE 1. Comparison of Numerical Integration of the Exact Solution (1) and (3) and the Hallén Filament Approximation (6), Weinstein Formula (19), and the SWK

	Exact Solution		Hallén Filament Approximation		Simple Averaging	
kz	Real	Imaginary	Real	Imaginary	Real	Imaginary
						0.001 (=ka)
0.001	0.47952347	−0.12612700	0.47952406	−0.12549869	0.4607	−0.1100
0.010	0.47848421	−0.11681319	0.47848441	−0.11679040	0.4612	−0.1089
0.100	0.47045314	−0.09447538	0.47045318	−0.09447468	0.4638	−0.0983
0.500	0.44979931	−0.06892960	0.44979932	−0.06892952	0.4514	−0.0679
1.000	0.43544673	−0.05833887	0.43544674	−0.05833884	0.4354	−0.0557
2.000	0.41905322	−0.04972731	0.41905322	−0.04972730	0.4176	−0.0474
5.000	0.39679600	−0.04164043	0.39679600	−0.04164043	0.3948	−0.0403
10.000	0.38075004	−0.03733794	0.38075004	−0.03733794	0.3788	−0.0364
						0.010 (=ka)
0.010	0.72067080	−0.28649826	0.72072689	−0.28021016	0.6640	−0.2354
0.100	0.70268351	−0.21351438	0.70269988	−0.21328330	0.6740	−0.2137
0.500	0.65901916	−0.14721664	0.65902299	−0.14719598	0.6574	−0.1446
1.000	0.62971087	−0.12134681	0.62971277	−0.12133866	0.6269	−0.1153
2.000	0.59700466	−0.10055131	0.59700555	−0.10054792	0.5923	−0.0951
5.000	0.55383346	−0.08099890	0.55383377	−0.08099778	0.5488	−0.0775
10.000	0.52360703	−0.07057560	0.52360716	−0.07057510	0.5190	−0.0681
						0.100 (=ka)
0.100	1.31231430	−0.75973227	1.31667978	−0.69527401	1.1343	−0.6711
0.500	1.17988440	−0.46218380	1.18179344	−0.45447198	1.1468	−0.4412
1.000	1.09721339	−0.36082412	1.09816573	−0.35805664	1.0748	−0.3346
2.000	1.00902834	−0.28270580	1.00945101	−0.28167250	0.9878	−0.2603
5.000	0.89858823	−0.21132588	0.89871995	−0.21102175	0.8803	−0.1968
10.000	0.82528611	−0.17439130	0.82533908	−0.17426576	0.8099	−0.1641

the Hallén filament approximation have a 10% error in the imaginary part. We can thus assume that the Hallén filament approximation typically has small error when applied to realistic antennas. The two averaging formulas have small errors, and the simple averaging is closest to the exact solution. The SWK interpolating formula gives a fairly accurate real part for this case. The Weinstein formula does not give very good numbers for this large ka.

Figures 2c and 2d show, for $ka = 0.1$ and $ka/10 \leq kz \leq 1000$, the real and imaginary parts of G, respectively, for all eight entries. For the useful range of $kz \geq ka$ the two averaging formulas are the closest to the exact solution and the Weinstein formula is the next closest. The SWK interpolating formula gives fairly accurate numbers for the real part. Figures 3a and 3b show, for $ka = 0.01$ and $ka/10 \leq kz \leq 1000$, the real and imaginary parts, respectively, for all eight entries. The two averaging formulas and the Weinstein formula are comparable in accuracy. The averaging formulas are more accurate than the Weinstein formula for kz somewhere between 0.1 and 1, and vice versa for kz not in this range. Figures 3c and 3d show, for $ka = 0.001$ and $ka/10 \leq kz \leq 1000$, the real and imaginary parts, respectively, for all eight entries. The same comment that applied to Figures 3a and 3b also applies to Figures 3c and 3d.

In conclusion, it is interesting to note that the simple averaging formula (10) is the simplest and also gives the most accurate approximation to the exact solution throughout the parameter range studied. The alternative averaging formula (12) gives almost as accurate numbers as the simple averaging. Next in order of accuracy is the Weinstein formula (19). The SWK interpolating formula (18) is somewhat less accurate, giving too large an imaginary part for G (the imaginary part of G is negative). For asymptotic formulas, the SWK is slightly more accurate than the Hallén first order, but both have very small errors when $kz \geq 0.5$. Finally, the Hallén filament approximation has essentially no error for practical antenna applications.

the Simple Averaging Formula (10), The Alternative Averaging Formula (12), SWK (17), the Hallén First-Order (16), the Interpolating Formula (18)

Alternative Averaging		SWK		Hallén First Order		Weinstein		SWK Interpolating	
Real	Imaginary	Real	Imaginary	Real	Imaginary	Real	Imaginary	Real	Imaginary
0.4608	−0.1096	0.8110	−0.1614	0.8520	−0.1905	0.4676	−0.1155	0.4600	−0.0526
0.4610	−0.1069	0.6347	−0.0997	0.6551	−0.1103	0.4680	−0.1126	0.4595	−0.0525
0.4586	−0.0926	0.5193	−0.0670	0.5306	−0.0717	0.4658	−0.0976	0.4544	−0.0514
0.4440	−0.0693	0.4601	−0.0527	0.4680	−0.0555	0.4509	−0.0728	0.4352	−0.0472
0.4314	−0.0585	0.4385	−0.0478	0.4453	−0.0502	0.4378	−0.0613	0.4199	−0.0439
0.4160	−0.0495	0.4187	−0.0437	0.4247	−0.0456	0.4218	−0.0517	0.4031	−0.0405
0.3944	−0.0412	0.3951	−0.0389	0.4002	−0.0405	0.3995	−0.0429	0.3815	−0.0363
0.3787	−0.0369	0.3790	−0.0358	0.3834	−0.0371	−0.3832	−0.0382	−0.3664	−0.0335
0.6646	−0.2309	1.1057	−0.2932	1.1982	−0.3987	0.6800	−0.2569	0.6784	−0.1137
0.6650	−0.1996	0.8110	−0.1614	0.8520	−0.1905	0.6830	−0.2224	0.6676	−0.1102
0.6415	−0.1456	0.6796	−0.1141	0.7045	−0.1282	0.6606	−0.1613	0.6282	−0.0977
0.6180	−0.1202	0.6347	−0.0997	0.6551	−0.1103	0.6359	−0.1323	0.5976	−0.0885
0.5887	−0.0991	0.5951	−0.0878	0.6121	−0.0960	0.6047	−0.1081	0.5647	−0.0791
0.5478	−0.0793	0.5495	−0.0750	0.5629	−0.0809	0.5610	−0.0855	0.5240	−0.0682
0.5186	−0.0690	0.5193	−0.0670	0.5306	−0.0717	0.5299	−0.0738	0.4964	−0.0612
1.1216	−0.6171	1.6726	−0.6080	1.8272	−1.1867	1.1048	−0.8058	1.1997	−0.3414
1.0981	−0.4329	1.2354	−0.3604	1.3540	−0.5280	1.1485	−0.5760	1.0875	−0.2841
1.0464	−0.3436	1.1057	−0.2932	1.1982	−0.3987	1.1095	−0.4492	1.0050	−0.2445
0.9760	−0.2696	0.9986	−0.2415	1.0704	−0.3105	1.0374	−0.3417	0.9205	−0.2065
0.8773	−0.2011	0.8831	−0.1906	0.9350	−0.2320	0.9268	−0.2440	0.8220	−0.1657
0.8088	−0.1661	0.8110	−0.1614	0.8520	−0.1905	0.8491	−0.1958	0.7587	−0.1417

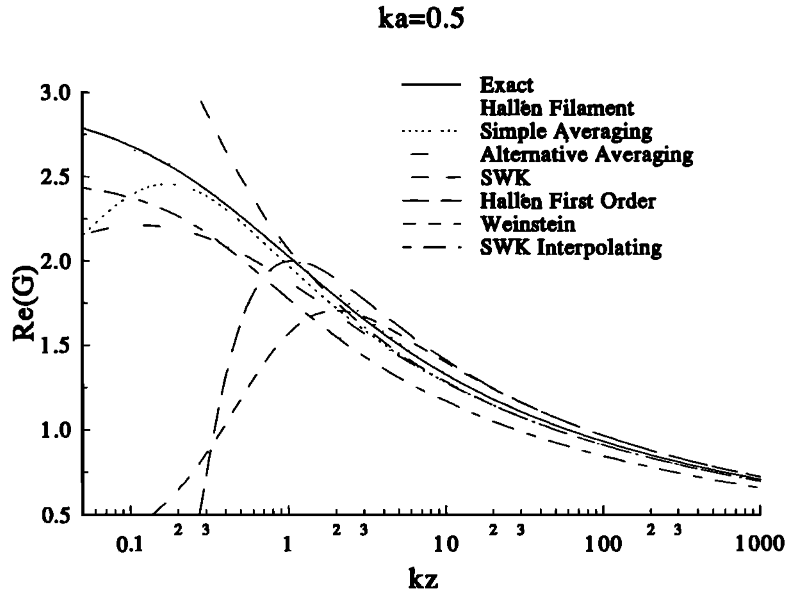

Fig. 2*a*. The real part of the antenna current normalized to give $G = I\eta_0/Ve^{ikz}$, which is (3).

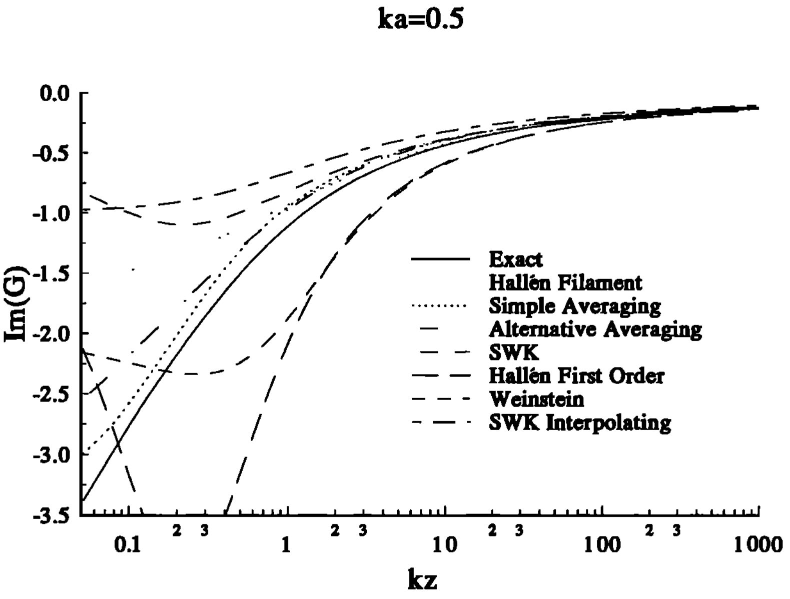

Fig. 2*b*. The imaginary part of the antenna current normalized by use of (3).

6. AN ASYMPTOTIC EXPANSION OF ANTENNA CURRENT

In this section we derive an asymptotic expansion of antenna current and relate our expansion to existing works. Although the finite antenna solution had previously been given by Hallén, let us comment that for infinite antennas the first zero-order formula for input impedance and the asymptotic formula, for $kz \rightarrow \infty$, together with proper interpretations of the meaning of these formulas, were apparently first given by

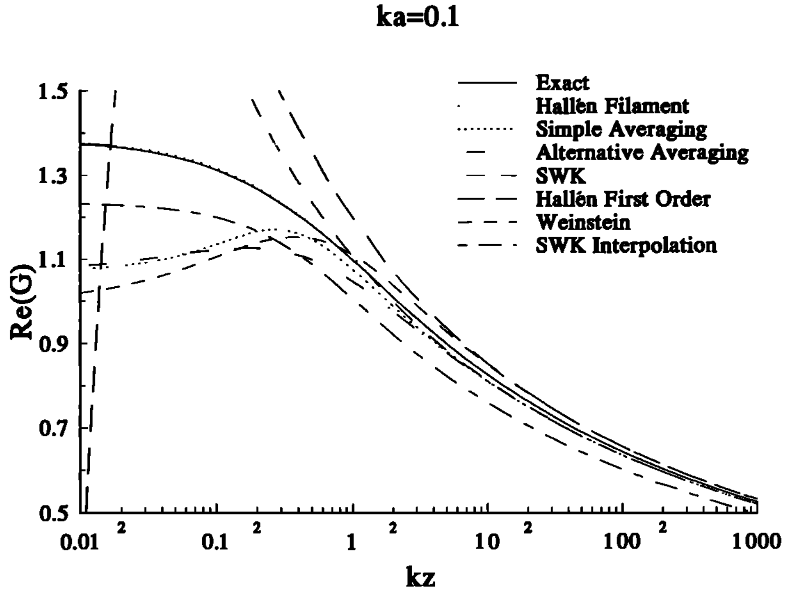

Fig. 2*c*. The real part of the antenna current normalized by use of (3).

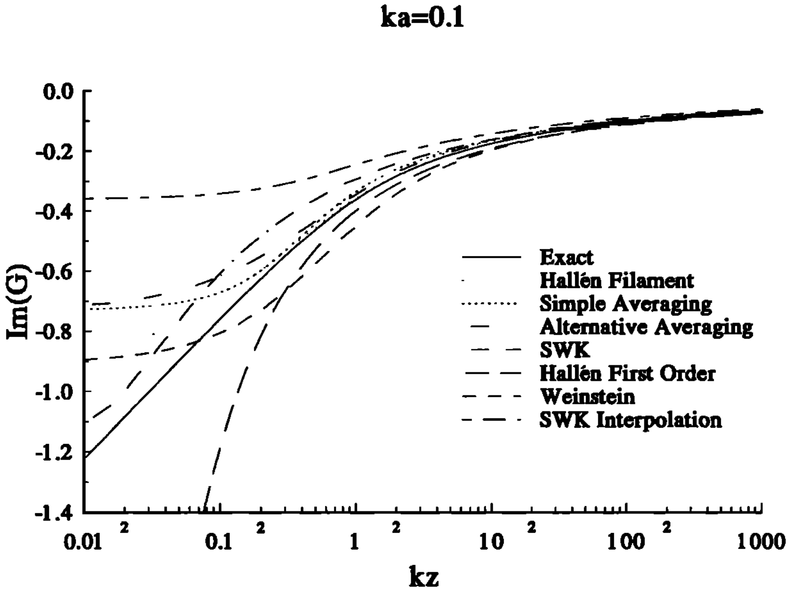

Fig. 2*d*. The imaginary part of the antenna current normalized by use of (3).

Vladimirskii [1944]. He also briefly discussed the logarithmic singularity of the gap capacitance.

In order to demonstrate the use of averaging for deriving an asymptotic series, we consider the limit as $kz \to \infty$. We can thus approximate $(1 + i\eta/2)$ in (8) by 1 and expand the logarithmic factor in the integrand in a Taylor's series in $\ln(\eta)$ at the expansion point x_0. When the expansion point is chosen as $x_0 = -\ln(kz) - \gamma$ given by (9), the second term of the series vanishes, and the resulting expansion for $G(\beta_p, kz)$ is

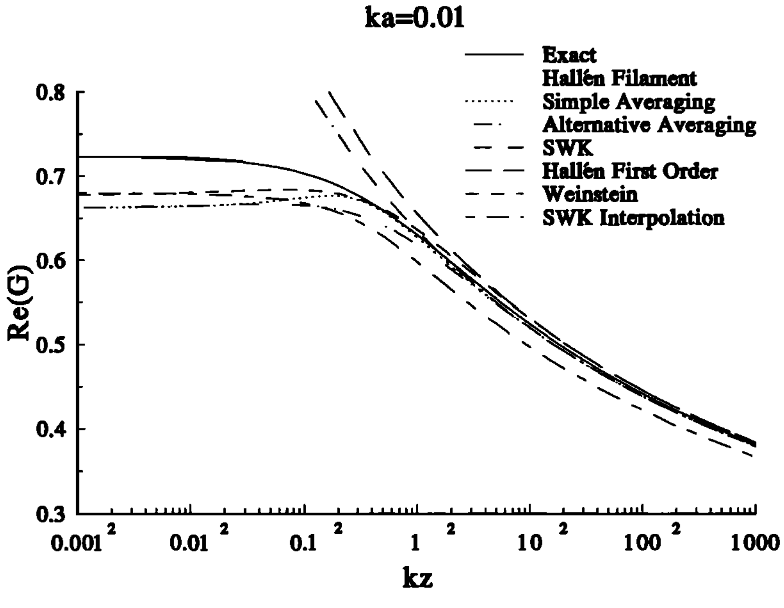

Fig. 3*a*. The real part of the antenna current normalized by use of (3).

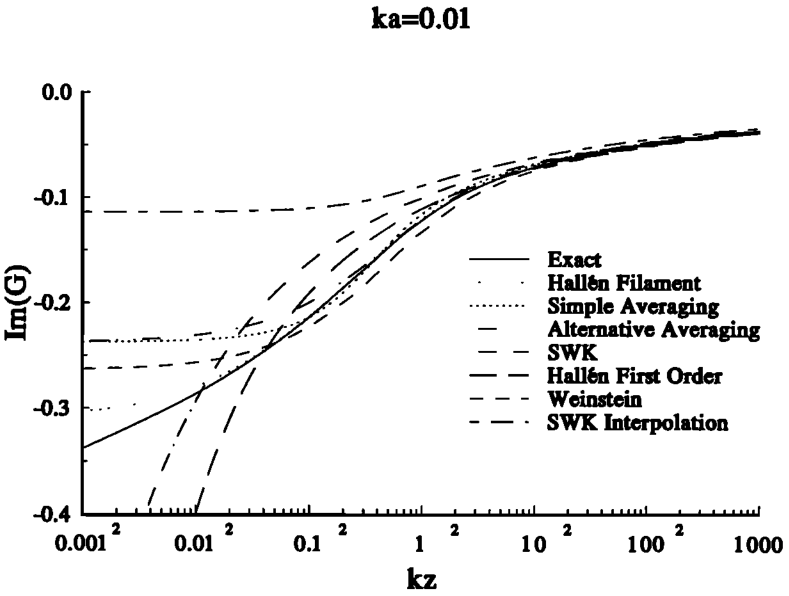

Fig. 3*b*. The imaginary part of the antenna current normalized by use of (3).

$$G(\beta_p, kz) \sim i\left(\ln\left[\frac{\Omega - i\pi}{\Omega + i\pi}\right] + \sum_{n=2}^{\infty} \frac{\rho_n}{n}\left[\frac{1}{(\Omega + i\pi)^n} - \frac{1}{(\Omega - i\pi)^n}\right]\right), \tag{26}$$

where

$$\rho_n = kz \int_0^{\infty} [\ln(\eta) + \ln(kz) + \gamma]^n e^{-kz\eta}\, d\eta. \tag{27}$$

The quantity ρ_n can also be determined by differentiating the following generating function:

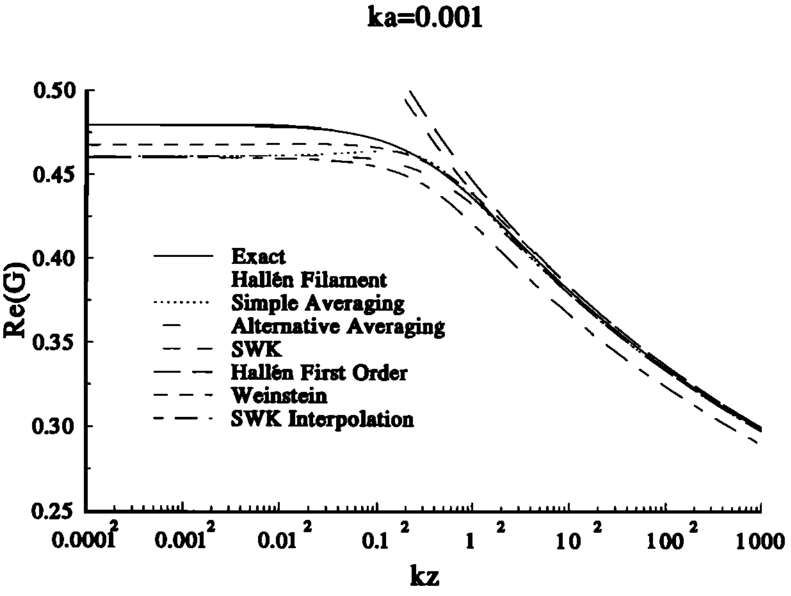

Fig. 3*c*. The real part of the antenna current normalized by use of (3).

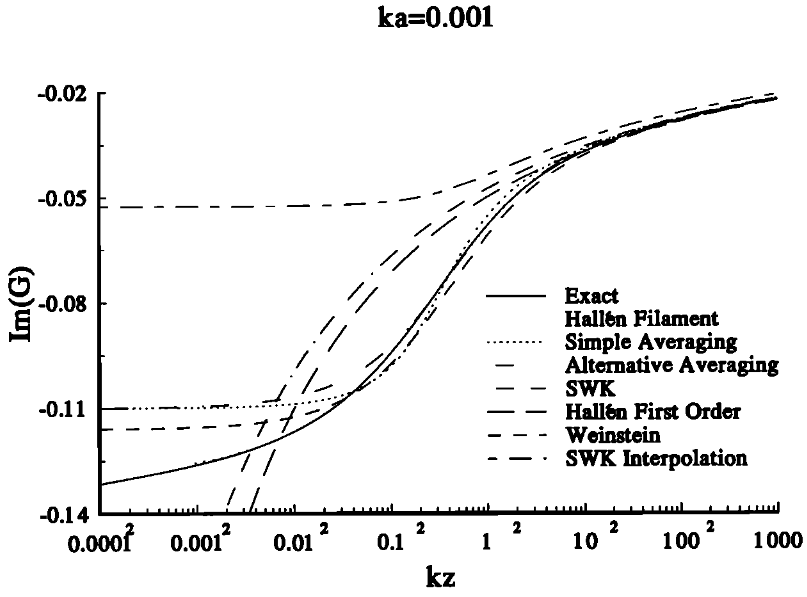

Fig. 3*d*. The imaginary part of the antenna current normalized by use of (3).

$$g(s) = kz\int_0^\infty (\eta kz e^{\gamma})^s e^{-kz\eta}\, d\eta = e^{\gamma s}\Gamma(1+s) = \sum_{n=0}^{\infty} \frac{\rho_n}{n!} s^n. \tag{28}$$

The leading coefficients are

$$\rho_0 = 1,$$

$$\rho_1 = 0,$$

$$\rho_2 = \zeta(2) = \pi^2/6 \approx 1.644934,$$

$$\rho_3 = -2\zeta(3) \approx -2.404114,$$

$$\rho_4 = 3\pi^4/20 \approx 14.611364,$$

$$\rho_5 = -24\zeta(5) - 10\pi^2/3\ \zeta(3) \approx -64.432353,$$

$$\rho_6 = 40[\zeta(3)]^2 + 61\pi^6/168 \approx 406.873470,$$

$$\rho_7 = -720\zeta(7) - 84\pi^2\zeta(5) - 21\pi^4/2\ \zeta(3)$$

$$\approx -2815.131424,$$

and

$$\rho_8 = 2688\zeta(3)\zeta(5) + 560\pi^2/3\,[\zeta(3)]^2 + 1261\pi^8/720$$

$$\approx 22630.60734.$$

The leading term in (26) was given in (17) and discussed before.

Kunz [1963] gave an asymptotic expansion of antenna current in terms of the sum of two series with one in inverse powers of Ω and another like series divided by kz. He pointed out that the first series is dominating and computed the coefficients of a few leading terms of both series. He also claimed that the leading term of his first series is more accurate than Hallén's first-order formula (16). It is clear that Kunz's leading term has a smaller absolute value in the denominator than that of Hallén's and thus would make the numerical value farther away from the numerical integration value of the exact integral. What Hallén did was to choose the large parameter such that the second term of the series vanishes, which is in the same spirit as the averaging method.

Hurd [1966] identified the general coefficient as essentially the *n*th derivative of the Gamma function evaluated at unity. Note that the transformation from this general coefficient to the coefficient of Kunz's series is not readily available. In the present asymptotic representation a similar general coefficient is used directly for the expansion, and thus the extra transformation has been eliminated through the choice of a large parameter.

Furthermore, the use of averaging eliminates the first-order term, and thus the leading term is good to the first order (two terms). The first term alone is a reasonable approximation even if kz is of the order of unity. Of course, to apply this result as an approximation to the original exact integral (1), the restriction of the Hallén filament approximation $z \gg a$ must still be observed. Numerical comparisons of (1) and (2) or (6) in the last section has relaxed this condition considerably.

Kuehl [1960] applied a saddle point integration to evaluate infinite antenna current for $kz \to \infty$. It is interesting to note that the nature of the integral (2) is such that no large parameter is available in the exponent, and thus one cannot approximate the integration along the steepest descent path by a Gaussian [*De Bruijn*, 1981], as was done by *Kuehl* [1960].

7. CONCLUSIONS

Two averaging formulas are derived. The simpler "simple averaging formula" (10) is slightly more accurate than the alternative averaging formula (12) when compared to the numerical integration of the exact solution (1) and (3). This simple averaging formula is useful for all practical applications. The alternative averaging formula is of theoretical interest in general antenna solutions. The Hallén filament approximation (6) is also compared to the exact solution and found to have typically small error when applied to realistic antennas. The Weinstein formula (19) has a slightly larger error than the two averaging formulas for a wide range of parameters. The SWK interpolating formula (18) has the largest error of all the uniform formulas but is still quite accurate. The two asymptotic formulas: the Hallén first order (16) and SWK (17) are accurate even when $kz \geq 0.5$; SWK is slightly more accurate than the Hallén first order. An asymptotic expansion of antenna current for $kz \to \infty$ is derived, and the leading term is shown to give rise to the SWK formula.

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